

# QUANTUM FILTERING OF MARKOV SIGNALS WITH WHITE QUANTUM NOISE

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**ABSTRACT.** Time-continuous non-anticipating quantum processes of nonde-molition measurements are introduced as the dynamical realizations of the causal quasi-measurements, which are described in this paper by the adapted operator-valued probability measures on the trajectory spaces of the generalized temporal observations in quantum open systems. In particular, the notion of physically realizable quantum filter is defined and the problem of its optimization to obtain the best a posteriori quantum state is considered. It is proved that the optimal filtering of a quantum Markovian Gaussian signal with the Gaussian white quantum noise is described as a coherent Markovian linear filter generalizing the classical Kalman filter. As an example, the problem of optimal measurement of complex amplitude for a quantum Markovian open oscillator, loaded to a quantum wave communication line, is considered and solved.

## 1. INTRODUCTION

At present, due to invention of laser as a coherent quantum generator, the dynamical problem of optimal reception of optical signals with quantum noise, taking into account the fundamental limitations caused by the quantum nature of the electromagnetic waves, is becoming actual. Quantum information and communication theory which has been developing so far (See recent review [1] and the references in there) is based on static (nontemporal), or single step (instantaneous) theory of quantum measurement, and quantum statistical inference does not take into account physical causality in due course of the dynamical propagation. As the result, the optimal estimators based on nontemporal quantum measurements are noncausal since they may depend at each intermediate time  $t$  not only on the past but also on future results of measurements. An attempt to develop the temporal, multistep variant of quantum measurement and decision theory to solve the problems of the dynamical filtering in discrete time was made in [2]. However, because of the very restrictive class of quantum measurement process as a sequences of independent, nonadaptive quantum measurements considered in this paper, even the simplest Gaussian problem for quantum linear filtering (quantum Kalman filtering) was not solved, with the exception of a degenerated case, and no time-continuous generalization is possible within this restricted method.

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*Date:* January 20, 1979.

*Key words and phrases.* Quantum white noise, Quantum output process, Quantum linear diffusion, Quantum adaptive measurement, Quantum Kalman filter.

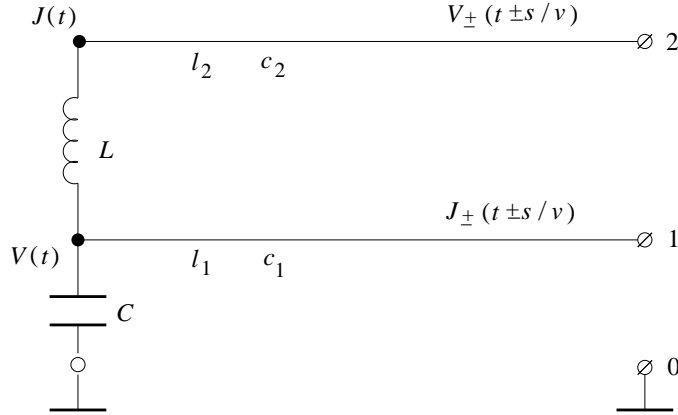
Translated from Russian by R.V. Belavkin for Proceedings of an International Workshop, held July 1994, in Nottingham, England: "Quantum Communications and Measurement", Plenum Press, New York and London, pp. 381 – 391. The original was published in: Radiotekhnika i Elektronika, 25 (1980) pp. 1445–1453.

Here we show, while solving the filtering problems for a quantum diffusion, that these difficulties can be avoided if a natural much wider class of temporal adaptive quantum measurements is considered, assuming their dependence not only on time, but also on the results of preceding measurements. Our dynamical model for the temporal quantum causal measurements has a natural time-continuous setup based on quantum stochastic differential equations. An appropriate quantum measurement device, that realizes the adaptive measurements, is called the nonanticipating (or causal) quantum filter with a memory. The optimization of estimation based on independent quantum measurements in the narrower class of filters without memory is in our treatment a problem with constraints leading to a poorer quality of filtration. As is proved below, the optimal time-continuous filtering of Markov Gaussian signals with the background quantum white noise is realized by a quantum linear filter based on the adaptive coherent quantum measurements. The optimal filter has a Markov memory such that the next optimal quantum measurement in general depends on the result of the present measurement but is independent of the preceding measurement results in this Markovian case. We show that such coherent quantum filter can be realized by indirect heterodyne measurements adding an independent vacuum quantum noise, and by causal processing of the temporal measurement results using a classical linear Kalman-Bucy filter [3]. A nonlinear (and quasi-linear) generalization of this result for the non-Gaussian case, which corresponds to the classical nonlinear filtering theory of Stratonovich [4], is also of our interest and will be considered elsewhere.

Let us first consider a physical model of quantum open system with output channel, in which the problem of optimal quantum filtering arises naturally, giving a solution of this problem. Then we shall set up a general rigorous formulation of the filtering problem as the problem of quantum optimal temporal estimation, and derive the solution of this problem in the Gaussian case.

## 2. OPTIMAL OBSERVATION OF QUANTUM OSCILLATOR AT THE OUTPUT OF WAVELINE

Let us consider an electromagnetic oscillator with frequency  $\Omega$  loaded on a pair of communication channels, the transmission lines which are described by wave conductivity  $G$  and wave resistance  $R$  respectively, as shown in the figure, where  $\Omega = 1/\sqrt{LC}$ ,  $G = \sqrt{c_1/l_1}$ ,  $R = \sqrt{l_2/c_2}$ , and propagation velocities of the waves  $v_i = 1/\sqrt{l_i c_i}$ ,  $i = 1, 2$  are assumed to be equal:  $v_1 = v_2 = v$ :



We consider the case when the lines are homogeneous and that the measuring device (receiver) which is set at their outputs of the lines  $(0 - 1)$  and  $(0 - 2)$  is ideally conjugated with these lines such that there is no reflection of the received information (i.e. incoming waves  $J_-, V_-$ ) into the radiated noise (i.e. outgoing waves  $J_+, V_+$ ) at the end of the lines. Such system is open, but it is entirely described by the dynamic variables of voltage  $V(t)$  and current  $J(t)$  on the contour  $(L, C)$ , and by the pairs of running waves

$$J_{\pm}(t \pm s/v) = J_1(s, t) \pm GV_1(s, t), \quad V_{\pm}(t \pm s/v) = V_2(s, t) \pm RJ_2(s, t)$$

of current and voltage in the first and second lines respectively as the solutions to the telegraph wave equations. The boundary conditions  $V_1(0, t) = V(t)$ ,  $J_2(0, t) = J(t)$  induce on the open oscillator the following pair of Langevin equations

$$\begin{aligned} C\dot{V}(t) + GV(t) - J(t) &= J_+(t), & L\dot{J}(t) + RJ(t) + V(t) &= V_+(t), \\ J_-(t) &= J_+(t) - 2GV(t), & V_-(t) &= V_+(t) - 2RJ(t), \end{aligned}$$

where the second pair of equations, corresponding to these boundary conditions, determines the output waves running from the oscillator to the receiver.

Assuming, for simplicity, that the following gage invariance condition  $R/L = G/C \equiv \gamma$  is fulfilled, the above system of equations can be written in the one-dimension complexified form

$$\dot{x}(t) + \left(i\Omega + \frac{1}{2}\gamma\right)x(t) = x_+(t), \quad x_-(t) = x_+(t) - \gamma x(t)$$

in terms of the complex amplitudes

$$\begin{aligned} x(t) &= \frac{1}{\sqrt{2}} \left( \sqrt{C}V(t) + i\sqrt{L}J(t) \right), \\ x_{\pm}(t) &= \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{C}}J_{\pm}(t) + \frac{i}{\sqrt{L}}V_{\pm}(t) \right). \end{aligned}$$

Now we move from the Langevin towards a stochastic description of the corresponding quantum system. In the classical case, assuming that the input noise  $x_+(t)$ , radiated by the receiver at the time  $t - s/v$ , is the equilibrium noise of the temperature  $T$ , we can treat it as a complex circular-invariant white noise with the intensity given by the Nyquist formula  $\sigma = kT$  (where  $k$  is the Boltzman constant). In the quantum case complex amplitude  $x(t)$  should be replaced by the corresponding operator  $\tilde{x}(t)$  in the Heisenberg picture, satisfying the canonical commutation relations

$$(2.1) \quad [\tilde{x}(t), \tilde{x}(t)^*] \equiv \tilde{x}(t)\tilde{x}(t)^* - \tilde{x}(t)^*\tilde{x}(t) = \hbar\check{1},$$

where  $\tilde{x}(t)^*$  is the Hermitian adjoint operator and  $\check{1}$  is the unit operator in the Hilbert space of quantum-mechanical states of this system ( $\hbar$  is the Planck constant,  $\tilde{x}(t)/\sqrt{\hbar} = \tilde{a}(t)$  is the annihilation operator for oscillator quanta at the time  $t$ ). By virtue of linearity the corresponding Heisenberg equation for  $\tilde{x}(t)$  should have the same form as for the classical complex amplitude  $x(t)$ , where the amplitude of the propagating wave  $x_+$  have to be replaced by the operator  $\tilde{x}_+(t)$  with the commutation relations ensuring preservation of the canonical commutation relations (2.1). This can be achieved if we take

$$(2.2) \quad [\tilde{x}_{\pm}(t), \tilde{x}_{\pm}(t')] = 0, \quad [\tilde{x}_{\pm}(t), \tilde{x}_{\pm}(t')^*] = \gamma\hbar\delta(t - t'),$$

where we have taken into account that the commutators for the output wave  $\tilde{x}_-(t)$  should coincide with the commutators of  $\tilde{x}_+(t)$  due to its linear relation with  $\tilde{x}_+(t)$ , as it is also assumed that  $\tilde{x}_+(t)$  commutes both with  $x(r)$ ,  $\tilde{x}_-(r)$  and with  $x(r)^*$ ,  $\tilde{x}_-(r)^*$  for  $r \leq t$ . As shown in [6], the above equations and commutation relations can be obtained by a canonical quantization of any open oscillator in the “rotating wave” representation and the narrow-band approximation. Introducing the notations

$$(2.3) \quad \tilde{v}(t) = \tilde{x}_+(t) = -\hat{v}(t), \quad \tilde{y}(t) = \tilde{x}_-(t - s/v) = -\hat{v}(t)$$

for the forward and backward waves at the input and output of the transmission line of length  $s$  respectively, let us write these equations in the following standard form

$$(2.4) \quad \frac{d}{dt} \tilde{x}(t) + \alpha \tilde{x}(t) = \tilde{v}(t), \quad \hat{y}(t + s/v) = \gamma \tilde{x}(t) + \hat{v}(t),$$

where  $\alpha = i\Omega + \gamma/2$ . The first equation, rewritten in terms of quantum stochastic differentials as

$$d\tilde{x}(t) + \alpha \tilde{x}(t)dt = d\tilde{v}_t,$$

where  $\tilde{v}_t = \int_0^t \tilde{v}(t)dt$ , can be easily integrated. The solution

$$\tilde{x}(t) = e^{-\alpha t} \tilde{x}(0) + \int_0^t e^{\alpha(r-t)} d\tilde{v}_r$$

does not depend on the output wave  $\hat{y}(t')$  with  $t' < t + s/v$  and it commutes with  $\hat{y}(t')$  as well as with  $\hat{y}(t')^*$  for all such  $t'$ . This commutativity, reflecting the ideal conjugacy between the output measurement device and the transmission lines, will play an important role for the causal prediction of  $\tilde{x}(t)$ , called the nondemolition condition. This condition, together with Markovianity of  $\tilde{x}(t)$  corresponding to the assumption that  $\tilde{v}(t)$  is a quantum white noise, which is fulfilled under the narrow-band approximation for the equilibrium state of any measurement device sending the quantum thermal noise wave  $\tilde{v}(t)$  into the line towards the oscillator, significantly simplifies the optimization problem of the continuous measurement in the quantum system under the consideration.

Let us now consider the problem of nondemolition observation of the amplitude operator  $\tilde{x}(t)$  of the quantum oscillator at the output of the quantum noisy channel described as above. One would like to obtain an optimal estimate  $x(t)$  of this amplitude as a classical stochastic complex-valued process, being observed in a real time on the output of the transmission line. The mean square optimization problem defining at each  $t$  such directly measurable nonanticipating estimate  $x(t)$  based on the results of the previous, in general indirect quantum nondemolition measurements, is the minimization problem of the mean quadratic error

$$(2.5) \quad \left\langle \left( \tilde{x}(t) - x(t) \right)^* \left( \tilde{x}(t) - x(t) \right) \right\rangle.$$

The continuous in time quantum nondemolition measurement is defined by any measurable non-anticipating transformation of the received quantum process  $\hat{y}(t)$  into the classical complex-valued one  $x(t)$ , which should be in general randomized in order to include also indirect nondemolition measurements of the noncommuting  $\hat{y}(t)^*$  and  $\hat{y}(t)$ . Identifying the classical process  $\{x(t)\}$  with a family of normal compatible operators in an extended Hilbert space, one can describe it at each  $t > 0$  by a quantum stochastic functional  $x(t, \cdot)$  of the past output operators

$\hat{y}(t')^*, \hat{y}(t'), t' \leq t$  and also of some additional independent quantum noises randomizing if necessary this estimate, with the operator value  $x(t)$  which commutes with all operators  $x(t')^*, x(t')$  even if  $t' \geq t$ . Note that such estimator  $x(t)$  will not necessary commute with all past estimated operators  $\tilde{x}(t'), \tilde{x}(t')^*$ , although due to the nondemolition property it will commute with them and all their future if  $t' \geq t - s/\nu$ . Such quantum directly observable process  $x(t)$ , which is the closest in the mean square sense (2.5) to the non-observed quantum stochastic process  $\tilde{x}(t)$ , can be considered as a dynamical realization of the optimal temporal quasi-measurement (or estimation) of the noncommuting  $\tilde{x}(t'), \tilde{x}(t')^*$ , based on the indirect observation of the given output process  $\hat{y}(t)$ .

In order to demonstrate such optimal temporal quasi-measurement let us consider the problem of quantum filtering in the Gaussian case when the initial quantum oscillator state is a circular-Gaussian with zero mathematical expectation  $\langle \tilde{x}(0) \rangle = 0$  and average number of quanta  $\Sigma \geq 0$ :

$$(2.6) \quad \langle \tilde{x}(0)^2 \rangle = 0, \quad \langle \tilde{x}(0)^* \tilde{x}(0) \rangle = \hbar \Sigma,$$

and the quantum noise  $\tilde{v}(t)$  is also circular-Gaussian white noise:  $\langle \tilde{v}(t) \rangle = 0$ ,

$$(2.7) \quad \langle \tilde{v}(t) \tilde{v}(t') \rangle = 0, \quad \langle \tilde{v}(t)^* \tilde{v}(t') \rangle = \hbar \nu \gamma \delta(t - t'),$$

where  $\nu = (\exp\{\hbar\gamma/kT\} - 1)^{-1}$  is the average number of quanta in an equilibrium state of the receiver with the temperature  $T$ . As will be shown in this paper, the optimal estimate  $x(t)$  of the operator  $\tilde{x}(t)$  minimizing the quadratic criterion (2.5) is given by a nonorthogonal Hermitian-positive Gaussian operator-valued measure which describes the coherent quasimeasurement of an integral nonanticipating transformation of the output process  $\hat{y}$ . It can be realized by the direct measurement of the classical complex random process  $x(t)$  described by the linear Langevin equation

$$(2.8) \quad \frac{d}{dt} x(t) + \alpha x(t) = \kappa(t) (y(t) - \gamma x(t))$$

with  $x(0) = \langle \tilde{x}(0) \rangle = 0$ . The input process  $y(t)$  in this equation is a classical complex (i.e. commutative normal) process given by the additive randomizing transformation  $y(t) = \hat{y}(t) + \hat{w}(t)$  of the received quantum process  $\hat{y}(t)$ , where  $\hat{w}(t)$  is the complex amplitude of an additional independent quantum noise with the opposite commutators

$$[\hat{w}(t), \hat{w}(t')] = 0, \quad [\hat{w}(t), \hat{w}(t')^*] = -\hbar \gamma \delta(t - t'),$$

zero expectations  $\langle \hat{w}(t) \rangle = 0$  and the minimal vacuum state correlations

$$(2.9) \quad \langle \hat{w}(t) \hat{w}(t') \rangle = 0, \quad \langle \hat{w}(t)^* \hat{w}(t') \rangle = \hbar \gamma \delta(t - t').$$

Since  $y(t)$  is equivalent to the classical  $\delta$ -correlated complex process of the intensity  $\sigma = \hbar(\nu + 1)\gamma$ , this Langevin equation can be identified with the classical Kalman-Bucy filter which is usually written in the stochastic differential form as

$$dx(t) + \alpha x(t)dt = \kappa(t) (dy_t - \gamma x(t)dt),$$

in terms of the continuous diffusive input process  $y_t = \int_0^t y(t)dt$ . In this equation  $\kappa(t) = (\Sigma(t) - \nu)/(1 + \nu)$ ,  $\Sigma(t)$  is the solution of the Riccati equation

$$(2.10) \quad \frac{d}{dt} \Sigma(t) = \frac{\gamma}{1 + \nu} (\nu - \Sigma(t)) (1 + \Sigma(t)),$$

with  $\Sigma(0) = \Sigma_0$  and the effective classical white noise intensity  $\sigma = \hbar\gamma(1 - \exp\{-\hbar\gamma/kT\})^{-1}$ . Thus the randomized nondemolition quasimeasurement of  $\tilde{x}(t)$  is realized by the direct observation of  $x(t)$  in the result  $y(t)$  of indirect temporal observation of  $\hat{y}(t)$  by the heterodyning processing  $\hat{y}(t) \mapsto \hat{y}(t) + \hat{w}(t)$  [7], where  $\hat{w}(t)$  plays the role of the reference quantum wave. The mean quadratic error (2.5) for optimal filtration in this case has the minimal value  $\hbar\Sigma(t)$ , which is not zero even under zero temperature  $T = 0$  corresponding to  $\sigma = \hbar\gamma$ , in the sharp contrast with the classical case when  $\sigma = 0$  at  $T = 0$ . This corresponds to adding into the measurement channel a vacuum quantum noise which makes possible the heterodyne indirect measurement of  $\hat{y}(t)$  statistically equivalent to a classical filtering of the complex white noise of the minimal intensity  $\hbar\gamma > 0$ . Note that this conclusion remains also valid for the quasi-classical oscillator corresponding to  $\Sigma_0 \gg 1$ , being indirectly observed in a quantum wave line. However in the infinite temperature case  $T \rightarrow \infty$  when  $\sigma \gg 1$ , the quantum consideration does not produce a substantial increase of the optimal filtering mean quadratic error compared with the one given by the classical Kalman-Bucy filter.

### 3. CAUSAL QUANTUM MEASUREMENT PROCESSES AND FILTERS

Since the sequences  $x(t_1), \dots, x(t_2)$  of the discrete-time measurements for arbitrary  $\{t_1 < \dots < t_n\}$  completely determine a continuous function  $x(t)$  under the temporal observation, the quantum measurement process can be statistically described by the multitime probability distributions  $\mathbf{P}(dx(t_1), \dots, dx(t_n))$ ,  $n = 1, 2, \dots$  which define the probability measure  $\mathbf{P}(d\{x(t)\})$  as a projective limit on the space of the trajectories  $\{x(t)\}$ . The statistical structure of quantum theory requires the probability measure to be a linear form with respect to the density operator  $\hat{\rho}$  of a quantum state, represented as

$$(3.1) \quad \mathbf{P}(d\{x(t)\}) = \text{Tr } \hat{\rho} \Pi(d\{x(t)\})$$

Here  $\text{Tr}$  means the Hilbert space trace,  $\Pi(d\{x(t)\}) \geq 0$  is a Hermitian-positive measure on the space of the observable trajectories  $\{x(t)\}$  with values in an output operator algebra  $\mathcal{B}$ . We call such  $\Pi$  the operator-valued probability measure (OPM) since it must be normalized to the identity operator  $\int \Pi(d\{x(t)\}) = \hat{1}$  due to the normalization of  $\mathbf{P}$ . This measure defines, in particular, the multitime OPM  $\Pi(dx(t_1) \dots dx(t_n))$  for any finite  $n = 1, 2, \dots$  which induce the probabilities of the time-discrete temporal observations as the linear functions of  $\hat{\rho}$  according to (3.1).

Every decomposition of the identity  $\hat{1} = \int \Pi(d\{x(t)\})$  into the positive operators  $\Pi(d\{x(t)\})$  corresponds, according to Naymark's theorem [8], to a measurement of some compatible set of operators, which are observed in an "extended quantum system". But only those measures  $\Pi$  correspond to the measurement processes physically realizable in real time, which satisfy the causality condition  $\Pi^t(dx^t) \in \mathcal{B}^t$ , where  $\Pi^t$  is the measure  $\Pi$  on the space of "reduced" realizations  $x^t = \{x(s)\}_{s \leq t}$ , and  $\{\mathcal{B}^t\}$ ,  $t \geq 0$  is a nondecreasing family of operator algebras  $\mathcal{B}^s \subseteq \mathcal{B}^t$ , defining the nondemolition observables up to the time instants  $t$ . Typically  $\mathcal{B}^t$  is generated by the subfamily  $\hat{b}^t = \{\hat{b}(s) : s \leq t\}$  of a given operator family  $\{\hat{b}(t)\}$ ,  $t \geq 0$ , describing an output quantum stochastic process  $\hat{b}(t)$  as in the example of first Section where  $\hat{b}(t) = \hat{y}(t)$ .

In order to have predictable behavior of a quantum object described by the Heisenberg operators  $\{\tilde{x}(t)\}$  generating an algebra  $\mathcal{A}$  under the indirect temporal

measurements in the algebra  $\mathcal{B}$ , the subalgebras  $\mathcal{B}^t$  must satisfy only the commutativity condition  $\mathcal{B}^t \subseteq \mathcal{A}'_t$  for each  $t$  with respect to the algebras  $\mathcal{A}_t$  generated by the present and future Heisenberg operators  $\tilde{x}_t = \{\tilde{x}(s) : s \geq t\}$  of the object such that there exist conditional expectation on  $\mathcal{A}_t \vee \mathcal{B}^t$  with respect to  $\mathcal{B}^t$  for each  $t$ . The measurement process should be also selfpredictable, for which we shall assume the existence of the conditional OPM  $\Pi_s^t(x^s, dx_s^t)$  on  $x_s^t = \{x(t')\}_{t' \in (s, t]}$ , describing by the formula (3.1) the non-anticipating processes of observation from any moment  $s < t$  up to  $t$  with known results  $x^s = \{x(t)\}_{t \leq s}$  of the preceding measurements. They should satisfy the compatibility condition

$$(3.2) \quad \Pi_{t_0}^{t_1}(x^{t_0}, dx_{t_0}^{t_1}) \Pi_{t_1}^{t_2}(x^{t_1}, dx_{t_1}^{t_2}) = \Pi_{t_0}^{t_2}(x^{t_0}, dx_{t_0}^{t_2})$$

for all  $t_0 < t_1 < t_2$  such that the Hermitian-positive operators  $\Pi_s^t(x^s, dx_s^t)^* = \Pi_s^t(x^s, dx_s^t)$  commute  $\Pi^s(dx^s) \in \mathcal{B}^s$ , having the values in the relative commutants  $\mathcal{B}_s^t$  of the output subalgebras  $\mathcal{B}^s$  with respect to  $\mathcal{B}^t$ . Such a compatible family  $\{\Pi_s^t\}_{s < t}$  of the conditional OPM  $\Pi_s^t(x^s, dx_s^t) \in \mathcal{B}_s^t$ , normalized to the identity operator  $\hat{1}$ , will be called *non-anticipating quantum filter*. The measures  $\Pi_s^t$ , which are independent of  $x^s$ , correspond to the filters without memory. The Markovian filters described by the conditional measures  $\Pi_s^t$ , which depend only on the last preceding value  $x(s)$ , are the simplest filters with the memory. In addition, the transition operator-valued measure  $\Pi_s^t(x(s), dx(t))$  defines, according to (3.2), the multitime conditional measures

$$\begin{aligned} & \Pi(x(t_0), dx(t_1) \cdots dx(t_{n-1}) dx(t_n)) \\ &= \Pi_{t_0}^{t_1}(x(t_0), dx(t_1)) \cdots \Pi_{t_{n-1}}^{t_n}(x(t_{n-1}), dx(t_n)), \end{aligned}$$

satisfying the operator-valued Smolukhowsky equation

$$(3.3) \quad \int_{X_{t_1}} \Pi_{t_0}^{t_1}(x(t_0), dx(t_1)) \Pi_{t_1}^{t_2}(x(t_1), dx(t_2)) = \Pi_{t_0}^{t_2}(x(t_0), dx(t_2)).$$

In the case of a linear measurable space  $X$ , for example a complex  $n$ -dimensional space  $\mathbb{C}^n$ , it is convenient to describe the Markovian filters by transitional operator-valued characteristic functions (OCF) defined as operator-valued Fourier integrals

$$(3.4) \quad \Phi_s^r(\mathbf{x}(s), \mathbf{u}(t)) = \int e^{i(\mathbf{u}(t)^+ \mathbf{x}(t) + \mathbf{x}(t)^+ \mathbf{u}(t))} \Pi_s^r(\mathbf{x}(s), d\mathbf{x}(t)),$$

where  $\mathbf{u}, \mathbf{x}$  are  $n$ -dimensional complex columns, and  $\mathbf{x}^+, \mathbf{u}^+$  are their conjugate rows. The normalization condition for  $\Pi_s^t$  gives  $\Phi_s^t(\mathbf{x}(s), 0) = \hat{1}$ , and the condition of Hermitian positivity  $\Pi_s^t \geq 0$  implies Hermitian positive-definiteness of the operator-matrix  $[\Phi(\mathbf{x}(s), \mathbf{u}_k - \mathbf{u}_l)] \geq 0$ , where  $\{\mathbf{u}_i, i = 1, 2, \dots\}$  is any finite collection of vectors  $\mathbf{u}_i \in \mathbb{C}^n$ . The operators  $\Phi_s^t$  commute on the non overlapping time intervals, and the Markovian condition (3.2) can be written in the following differential operator form

$$(3.5) \quad \Phi_{t_1}^{t_2} \left( -i \frac{\partial}{\partial \mathbf{u}(t_1)}, \mathbf{u}(t_2) \right) \Phi_{t_0}^{t_1}(\mathbf{x}(t_0), \mathbf{u}(t_1))|_{\mathbf{u}(t_1)=0} = \Phi_{t_0}^{t_2}(\mathbf{x}(t_0), \mathbf{u}(t_2)).$$

Under certain regularity condition it can be proved that the opposite is also true: for every family of regular operator-valued functions  $\{\Phi_s^t, s < t\}$ , which are continuous at  $u(t) = 0$ , satisfying the above normalization, Hermitian positive-definiteness and Markovianity condition (3.5), there exists unique Markovian filter having the OCF  $\Phi_s^t$ .

As an example of a Markovian filter we can take a linear coherent filter which is defined by OCF function

$$(3.6) \quad \Phi_s^t(\mathbf{x}(s), \mathbf{u}(t)) = e^{i\mathbf{u}(t)^+ \hat{\mathbf{x}}(t)} e^{i\hat{\mathbf{x}}(t)^+ \mathbf{u}(t)},$$

where  $\hat{\mathbf{x}}(t)$  is vector-column composed of the operators  $\hat{x}_1(t), \dots, \hat{x}_n(t)$  and  $\hat{\mathbf{x}}(t)^+$  is vector-row composed of the adjoint operators  $\hat{x}_1(t)^*, \dots, \hat{x}_n(t)^*$ . The operators  $\hat{\mathbf{x}}(t)$  entering into this anti-normal ordered expression for  $\Phi_s^t$  are assumed to satisfy the following linear quantum stochastic differential equation

$$(3.7) \quad \frac{d}{dt} \hat{\mathbf{x}}(t) + B(t) \hat{\mathbf{x}}(t) = K(t) \hat{\mathbf{b}}(t), \quad \hat{\mathbf{x}}(s) = \mathbf{x}(s)$$

with  $\hat{\mathbf{b}}(t)$  defined as  $m$ -dimensional column composed of the annihilation operators  $\hat{b}_1(t), \dots, \hat{b}_m(t)$  of the quantum output process satisfying the following canonical commutation relations

$$[\hat{b}_k(t), \hat{b}_l(t')] = 0, \quad [\hat{b}_k(t), \hat{b}_l^*(t')] = \delta_{kl} \delta(t - t').$$

Here  $B(t) = [\beta_{kl}(t)]$ ,  $K(t) = [\kappa_{kl}(t)]$  are complex matrices of the size  $n \times n$  and  $n \times m$  respectively, which may continuously depend on time. It is not hard to verify by integrating the equation (3.7) that the characteristic operator-function (3.6) satisfies all the above mentioned properties, including (3.5). In fact it can be easily seen that, since the operators  $\hat{\mathbf{x}}(r)$  are linear transformations from the annihilation ones  $\hat{\mathbf{b}}(t)$ , that (3.6) is the characteristic operator-function of the operator-valued Dirac  $\delta$ -measure

$$\Pi_s^t(\mathbf{x}(s), d\mathbf{x}(t)) = \mathcal{N}_- \{ \delta(\hat{\mathbf{x}}(t), d\mathbf{x}(t)) \},$$

which corresponds to the anti-normal ordering “ $\mathcal{N}_-$ ”, when the creation operators  $\hat{\mathbf{b}}^*(t)$  act to the left before  $\hat{\mathbf{b}}(t)$ . It is well known that such measure is generated by the coherent projectors and the Lebesgue measure on the complex space  $\mathbb{C}^n$ . For example, for the case of a fixed  $\mathbf{x}(0) = \mathbf{0}$  the measure  $\Pi(t, d\mathbf{x}) \equiv \Pi_0^t(\mathbf{0}, d\mathbf{x})$  has the form

$$(3.8) \quad \Pi(t, d\mathbf{x}) = |t, \mathbf{x}\rangle \langle t, \mathbf{x}| d\lambda(t, \mathbf{x}),$$

where  $|t, \mathbf{x}\rangle$  are the normalized right eigen vectors

$$(3.9) \quad \hat{\mathbf{x}}(t) |t, \mathbf{x}\rangle = \mathbf{x} |t, \mathbf{x}\rangle, \quad \mathbf{x} \in \mathbb{C}^n$$

for the operators  $\hat{\mathbf{x}}(t)$  defined by the equation (3.7) with the initial condition  $\hat{\mathbf{x}}(0) = \mathbf{0}$ . The element  $d\lambda$  of the volume in  $\mathbb{C}^n$ , normalizing the expression (3.8), is given by

$$d\lambda(t, \mathbf{x}) = \frac{1}{\det [\pi C(t)]} \prod_{i=1}^n d\operatorname{Re} x_i d\operatorname{Re} x_i,$$

where  $C(t)$  is the Hermitian matrix of commutators  $[\hat{x}_k(t), \hat{x}_l(t)^*] = (C(t))_{kl}$  satisfying the equation

$$\frac{d}{dt} C(t) + B(t) C(t) + C(t) B(t)^+ = K(t) K(t)^+$$

with  $C(0) = 0$  corresponding to  $\hat{\mathbf{x}}(0) = \mathbf{0}$ .



A physical realization of the above linear coherent quantum Markovian filter as optimal quasi-measurement of the operators  $\{\hat{\mathbf{x}}(t)\}$  is given by a precise direct measurement of the Markovian process  $\mathbf{x}(t)$  described by the linear classical stochastic differential equation

$$(3.10) \quad \frac{d}{dt}\mathbf{x}(t) + B(t)\mathbf{x}(t) = K(t)\mathbf{b}(t), \quad \mathbf{b}(t) = \hat{\mathbf{b}}(t) + \hat{\mathbf{c}}(t).$$

Here  $\hat{\mathbf{c}}(t)$  is independent of  $\hat{\mathbf{b}}(t)$  quantum “vacuum” vector noise with the opposite commutators

$$[\hat{c}_k(t), \hat{c}_l(t')] = 0, \quad [\hat{c}_k(t), \hat{c}_l(t')^*] = -\delta_{kl}\delta(t - t'),$$

which make the components of the sum  $\mathbf{b}(t)$  commuting and also commuting with the components of  $\mathbf{b}(t)^*$ , and  $\langle \hat{c}(t) \rangle = 0$ ,

$$\langle \hat{c}_k(t)\hat{c}_l(t') \rangle = 0, \quad \langle \hat{c}_k(t)^*\hat{c}_l(t') \rangle = \delta_{kl}\delta(t - t'),$$

and so  $\langle \hat{c}_k(t)\hat{c}_l(t')^* \rangle = 0$ . Therefore the momenta of the process  $\mathbf{x}(t)$  coincide with the anti-normal momenta for  $\hat{\mathbf{x}}(t)$ , in particular  $\langle \hat{\mathbf{x}}(t) \rangle = \langle \mathbf{x}(t) \rangle$ ,

$$\langle \hat{x}_k(t)\hat{x}_l(t) \rangle = \langle x_k(t)x_l(t) \rangle, \quad \langle \hat{x}_k(t)\hat{x}_l(t)^* \rangle = \langle x_k(t)x_l(t)^* \rangle.$$

#### 4. QUANTUM GAUSSIAN DIFFUSION AND LINEAR FILTERING

Let  $\check{\mathbf{x}}(t)$  be an  $n$ -dimensional quantum diffusive process with zero initial expectation  $\langle \check{\mathbf{x}}(0) \rangle = 0$  and the initial correlations

$$(4.1) \quad \langle \check{x}_k(0)\check{x}_l(0) \rangle = 0, \quad \langle \check{x}_k(0)^*\check{x}_l(0) \rangle = (R_0)_{lk},$$

where  $R_0$  is a given positive-definite matrix. The process is defined by the stochastic equation

$$(4.2) \quad \frac{d}{dt}\check{\mathbf{x}}(t) + A(t)\check{\mathbf{x}}(t) = J(t)\check{\mathbf{a}}(t),$$

where  $A(t)$ ,  $J(t)$  are matrices of size  $n \times n$  and  $n \times m$  respectively, and  $\check{\mathbf{a}}(t)$  is white quantum noise satisfying canonical commutation relations

$$(4.3) \quad [\check{a}_k(t), \check{a}_l(t')] = 0, \quad [\check{a}_k(t), \check{a}_l(t')^*] = \delta_{kl}\delta(t - t')$$

with zero expectations  $\langle \check{\mathbf{a}}(t) \rangle = 0$  and nonzero normal correlation matrix  $Q(t)$ :

$$(4.4) \quad \langle \check{a}_k(t)\check{a}_l(t') \rangle = 0, \quad \langle \check{a}_k(t)^*\check{a}_l(t') \rangle = (Q(t))_{lk}\delta(t - t').$$

We consider also an output system described at the output of a quantum linear noisy channel

$$(4.5) \quad \hat{\mathbf{b}}(t) = F(t)\check{\mathbf{x}}(t) + \hat{\mathbf{a}}(t)$$

by  $m \times n$  matrix  $F(t)$  and by  $m$ -dimensional quantum white noise  $\hat{\mathbf{a}}(t)$  with canonical commutators

$$(4.6) \quad [\hat{a}_k(t), \hat{a}_l(t')] = 0, \quad [\hat{a}_k(t), \hat{a}_l(t')^*] = \delta_{kl}\delta(t - t'),$$

zero expectations  $\langle \hat{\mathbf{a}}(t) \rangle = 0$  and a nonzero normal correlation matrix  $N(t)$ :

$$(4.7) \quad \langle \hat{a}_k(t)\hat{a}_l(t') \rangle = 0, \quad \langle \hat{a}_k(t)^*\hat{a}_l(t') \rangle = (N(t))_{lk}\delta(t - t').$$

The pair  $(\check{\mathbf{a}}(t), \hat{\mathbf{a}}(t))$  is assumed to be independent of  $\check{\mathbf{x}}(0)$  at least in the wide sense, having zero correlations with  $\check{\mathbf{x}}(0)^*$ , but it can have nonzero matrix  $T(t)$  of the mutual normal correlations in

$$(4.8) \quad \langle \check{a}_k(t)\hat{a}_l(t') \rangle = 0, \quad \langle \check{a}_k(t)^*\hat{a}_l(t') \rangle = (T(t))_{kl}\delta(t - t').$$

This is because we cannot assume them independent but satisfying the commutation relations

$$(4.9) \quad [\tilde{a}_k(t), \hat{a}_l(t')] = 0, \quad [\tilde{a}_k(t), \hat{a}_l(t')^*] = (D(t))_{kl} \delta(t - t'),$$

which are necessary for the nondemolition condition of mutual commutativity of all components  $\hat{\mathbf{b}}(t')$  with  $\tilde{\mathbf{x}}(t)^*$  for any  $t \geq t'$ . If  $\tilde{\mathbf{x}}(t)$  satisfies the commutation relations

$$(4.10) \quad [\tilde{x}_k(t), \tilde{x}_l(t)] = 0, \quad [\tilde{x}_k(t), \tilde{x}_l(t)^*] = (C(t))_{kl},$$

the matrix  $D$  is defined by the nondemolition condition as a solution to the equation  $JD + CF^+ = 0$  at each  $t$  (As it occurred in the example considered in the Section 2.) Usually the commutation relations  $C(t)$  of the object are preserved,  $C(t) = C_0$ , i.e.  $AC_0 + C_0A^+ = JJ^+$ . Note, however, that the nonzero commutation relations in (4.3) are not necessary for this preservation and for the nondemolition property

$$[\tilde{x}_k(t), \hat{y}_l(t')] = 0, \quad [\tilde{x}_k(t), \hat{b}_l(t')^*] = 0 \quad \forall t \geq t'$$

if the process  $\tilde{\mathbf{x}}(t)$  has the degenerate matrix  $C_0$ , e.g. if it is classical:  $C_0 = 0$ , in which case also  $D = 0$  for the nondegenerate  $J$ .

The problem of optimal nonstationary quantum filtering consist of finding a nonanticipating physically realizable measurement process, denoted by  $\mathbf{x}(t)$ , at the output of the quantum channel, minimizing at each time  $t$  the mean of the quadratic error

$$(4.11) \quad |\tilde{\mathbf{x}}(t) - \mathbf{x}(t)|^2 := \sum_{i=1}^n (\tilde{x}_i(t) - x_i(t))^* (\tilde{x}_i(t) - x_i(t)).$$

**Theorem 1.** *Let the initial quantum vector  $\tilde{\mathbf{x}}(0)$  be Gaussian with zero mathematical expectation and the correlations (4.1), the pair  $(\tilde{\mathbf{a}}(t), \hat{\mathbf{a}}(t))$  be also Gaussian with zero expectations and with correlations (4.4), (4.7), (4.8), and  $\hat{\mathbf{a}}(t)$  satisfy the canonical commutation relations (4.6). Then the optimal by the criterion (4.11) quantum filter for the quantum Markov Gaussian process  $\tilde{\mathbf{x}}(t)$ , defined by the linear quantum stochastic equation (4.2), is the Markovian coherent filter (3.6), described by the linear quantum stochastic equation (3.7), where  $\mathbf{x}(0) = 0$  and*

$$(4.12) \quad B = A + KF, \quad K = (PF^+ + JT^+)(N + I)^{-1}.$$

Here  $I$  is the identity  $m \times m$ -matrix, and  $P = P(t)$  is the a posteriori correlation matrix, which is given by the solution of the Riccati equation

$$(4.13) \quad \frac{d}{dt}P + AP + PA^+ + (PF^+ + JT^+)(N + I)^{-1}(FP + TJ) = JQJ^+$$

with the initial condition  $P(0) = R_0$ . The minimal expectation value of the quadratic error (4.11) at the time  $t$  is defined by the trace of  $P(t)$ . (The dependence of all the matrices in (4.12), (4.13) on  $t$  is omitted for brevity.)

PROOF. Given the fact that the criterion for the point filtering (4.11) depends on the value  $\mathbf{x}(t)$  of the observation  $\{\mathbf{x}(t)\}$  only at the last instant  $t$ , the average value of the quadratic error is defined only by the single-time operator-valued measure  $\Pi(t, d\mathbf{x}(t))$ . Therefore one can find the solution  $\Pi^0(t, d\mathbf{x}(t))$  of the statistical problem [10] of the optimal observation independently for each  $t$ . If it proves to be a single-time operator-valued measure for some physically realizable measurement process  $\Pi^0(d\{\mathbf{x}(t)\})$  in the sense of the Section 3, then the problem of optimal filtering will be solved.

Using the stochastic calculus for normal ordered quantum Gaussian variables, which was developed in [10], one can obtain the following representation for the average of the quadratic error (4.11) as a function of the operator-valued measure  $\Pi(t, d\mathbf{x})$ :

$$(4.14) \quad \langle |\tilde{\mathbf{x}}(t) - \mathbf{x}(t)|^2 \rangle = \sum_{i=1}^n (P(t))_{ii} + \int \text{Tr } \hat{R}(t, \mathbf{x}) \Pi(t, d\mathbf{x}).$$

Here  $P(t)$  is a matrix satisfying the Riccati equation (4.13),  $\hat{R}(t, \mathbf{x})$  is the operator-valued function

$$(4.15) \quad \hat{R}(t, \mathbf{x}) = \sum_{i=1}^n (\hat{x}_i(t) - x_i)^* \hat{\varrho}(t) (\hat{x}_i(t) - x_i),$$

of the operators  $\hat{\mathbf{x}}(t)$  satisfying the equation (3.7) defined by the matrices (4.12) with the initial condition  $\hat{\mathbf{x}}(0) = 0$ , and  $\hat{\varrho}(t)$  is a Gaussian density operator for the output process  $\hat{\mathbf{x}}(t)$ . This  $\hat{\varrho}(t)$  can be represented in the form of a normally ordered expression

$$\hat{\varrho}(t) = \det(G(R - P)^{-1}) \mathcal{N} \left[ \exp \left\{ - \sum_{k,l} \hat{x}_k^*(t) (R - P)^{-1}_{kl} \hat{x}_l(t) \right\} \right],$$

where  $G(t)$ ,  $R(t)$  are the solutions of the linear equations

$$\frac{d}{dt}G + BG + GB^+ = KK^+, \quad \frac{d}{dt}R + AR + RA^+ = JQJ^+$$

with the initial conditions  $G(0) = 0$  and  $R(0) = R_0$  respectively.

The second element of the right-hand side in (4.14) is not negative because of the Hermitian positivity of the operator (4.15), and it is equal to zero on the coherent measure (3.8) defined by equation (3.9) for  $\mathbf{x}_0 = 0$ . Hence, it is clear that the coherent measurement, described by this measure, is optimal for all  $t$ . Therefore, the coherent observation described by the OCF  $\Phi(t, \mathbf{u}) = \Phi_0^t(\mathbf{0}, \mathbf{u})$  in (3.6) with the initial condition  $\hat{\mathbf{x}}(0) = 0$  is the optimal measurement process. The Theorem is proved.

Note that the a posteriori state for the quantum diffusion  $\tilde{\mathbf{x}}(t)$  is described by the Gaussian stochastic density operator

$$\tilde{\varrho}(t) = \det(G(P + G)^{-1}) \mathcal{N} \left[ \exp \left\{ - \sum (\tilde{x}_k(t) - x_k(t))^* (P + G)^{-1}_{kl} (\tilde{x}_l - x_l(t)) \right\} \right]$$

given by the solution  $\mathbf{x}(t)$  of the classical stochastic diffusion equation (3.10) with  $\mathbf{x}(0) = 0$ .

As example, let us consider the  $n$ -dimensional quantum oscillator (4.2) that has  $\tilde{\mathbf{a}}(t) = \hat{\mathbf{a}}(t)$ ,  $JJ^+ = \hbar(A + A^+)$ , such that the commutators

$$(4.16) \quad [\tilde{x}_k, \tilde{x}_l] = 0, \quad [\tilde{x}_k, \tilde{x}_l^*] = \hbar \delta_{kl}$$

are preserved. The matching  $m$ -dimensional quantum communication line is described by the output wave

$$(4.17) \quad \hat{\mathbf{y}}(t) = G(t)\tilde{\mathbf{x}}(t) - J(t)\hat{\mathbf{a}}(t).$$

This wave commutes for  $G = A + A^+$  with  $\tilde{\mathbf{x}}(t')^*$ ,  $t' \geq t$ . In the former dimensionless units this line is described by the wave (4.5), where  $F = -J^+/\hbar$ , so that  $\hat{\mathbf{y}}(t) = \hbar F^+ \hat{\mathbf{b}}(t)$ . Let the correlation matrix  $N$  of the quantum channel noise  $\hat{\mathbf{a}}(t)$  be

proportional to the identity matrix:  $N = \nu I$ . Then the equation (4.13) for the dimensionless a posteriori correlation matrix  $S = P/\hbar$  will have the form

$$(4.18) \quad \dot{S} = i[S, H] - \frac{1}{2(\nu + 1)} ((S + I)G(S - \nu I) + (S - \nu I)G(S + I)),$$

where  $H = (A - A^+)/2i$ , and  $K(t) = (\nu I - S)J/(1 + \nu)$ . In particular, in the one-dimensional case  $m = n = 1$ , denoting  $S = \Sigma$ ,  $G = \gamma$ , we obtain (2.10). It is interesting to note that the stationary solution  $S = \nu I$  of the equation (4.18) corresponds to a nonstochastic optimal filtering which does not require at all any observation since in this case the amplification coefficient  $K(t)$  is zero. The initial condition  $S_0 = \nu I$  for the stationary solution  $S(t) = S_0$  of the Riccati equation (4.18) in the case  $H = \gamma I$  can be interpreted as a condition of thermal equilibrium for the initial statistical state of the oscillator having  $H = \Omega$  in the one dimensional case, and the quantum channel noise equilibrium state as it corresponds to equality of the temperature for the equilibrium state in the communication line and of the temperature  $T = \hbar\Omega/k \ln(1 + \Sigma^{-1})$  for the oscillator initial Gaussian with  $\Sigma_0 = \gamma$ .

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